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More discrete copies of $\mathbb{Z}$ in $\beta\mathbb{N}$

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ABSTRACT

In a previous paper we established that if q is any minimal idempotent in $\beta\mathbb{N}$, then for all except possibly one $p \in c\ell\{2^n\colon n \in \mathbb{N}\} \setminus \mathbb{N}$, $q + p + q$ generates an infinite discrete group. Responding to a question of Wis Comfort, we extend this result in two directions. We show on the one hand that for a minimal idempotent q, there is at most one prime r for which there exists $p \in c\ell\{r^n\colon n \in \mathbb{N}\} \setminus \mathbb{N}$ such that the group generated by $q + p + q$ is not both infinite and discrete. On the other hand, we show that for any $p \in \beta\mathbb{N}$, if $p \in c\ell(n\mathbb{N})$ for infinitely many $n \in \mathbb{N}$, then there is some minimal idempotent q such that the group generated by $q + p + q$ is infinite and discrete. We also show that if G is a countable discrete group and if p is a right cancelable element of G^*, then there is an idempotent $q \in G^*$ such that $q \cdot p \cdot q$ generates a discrete copy of $\mathbb{Z}$ in G^*.

We do not know whether there exist any minimal idempotent q and any p with $p \in c\ell(n\mathbb{N})$ for infinitely many $n \in \mathbb{N}$ such that the group generated by $q + p + q$ is not discrete. We show that if such a "bad" q exists, then there are many of them.

1. Introduction

For any discrete semigroup $(S, \cdot)$, the Stone–Čech compactification βS of S admits an extension of the operation so that $(\beta S, \cdot)$ is a compact right topological semigroup with S contained in its topological center. That is, for each $p \in \beta S$ the function ρ_p is continuous and for each $x \in S$, the function λ_x is continuous, where for $y \in \beta S$, $\rho_p(y) = y \cdot p$ and $\lambda_x(y) = x \cdot y$. We take the points of βS to be the ultrafilters on S, identifying the principal ultrafilters with the points of S. For an elementary introduction to the Stone–Čech compactification of a discrete semigroup, see [3].

If S is a semigroup, $E(S)$ will denote the set of idempotents in S. Any compact Hausdorff right topological semigroup T has a smallest two sided ideal $K(T)$, which is the union of all of the minimal right ideals of T and is also the union of all of the minimal left ideals of T. The intersection of any minimal left ideal and any minimal right ideal is a group and any two such groups are isomorphic. In particular, the set $E(K(T))$ of idempotents in $K(T)$ is not empty. Such idempotents are said to be *minimal*. They are precisely the idempotents that are minimal with respect to the ordering of idempotents under which $q \leqslant p$ if and only if $q = p \cdot q = q \cdot p$. If q is a minimal idempotent in T and is a member of the minimal left ideal L and the minimal right ideal R, then the group $L \cap R = q \cdot T \cdot q$. Given any idempotent $q \in T$,

$$H(q) = \bigcup\{G \subseteq T\colon G \text{ is a group with identity } q\}.$$

Then $H(q)$ is a group and $H(q) = q \cdot T \cdot q$ if and only if q is minimal. (See [2] or [3] for the facts in this paragraph.)

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We are concerned in this paper primarily with the semigroup $(\beta\mathbb{N},+)$, where $\mathbb{N}$ is the set of positive integers. (We write ω for the set of nonnegative integers.) In this semigroup if q is any idempotent, not just a minimal idempotent, then $\mathbb{Z}+q$ is an algebraic copy of the group $(\mathbb{Z},+)$. Since $q=q+q\in\beta\mathbb{N}+q=c\ell(\mathbb{N}+q)$, this group is not discrete. In [4] we showed that each maximal group in $K(\beta\mathbb{N})$ contains discrete algebraic copies of $\mathbb{Z}$ (as well as discrete copies of the free group on two generators).

Theorem 1.1. *Let $q\in E(K(\beta\mathbb{N}))$. For all but at most one $p\in c\ell\{2^n\colon n\in\mathbb{N}\}^*=c\ell\{2^n\colon n\in\mathbb{N}\}\setminus\mathbb{N}$, the group generated by $q+p+q$ is infinite and discrete.*

Proof. [4, Corollary 2.2]. □

As we remarked at the time, we found it inconceivable that an exceptional p in Theorem 1.1 could exist, but we could not (and still cannot) prove that there are none.

A particular subsemigroup of $\beta\mathbb{N}$ that arises in many contexts is $\mathbb{H}=\bigcap_{n=1}^{\infty}c\ell(2^n\mathbb{N})$. We observed that the discrete groups produced in Theorem 1.1 are contained in $\mathbb{H}$, and we asked [4, Question 3.8] whether there is a discrete copy of $\mathbb{Z}$ in $\beta\mathbb{N}$ which is not contained in $\mathbb{H}$. We were presenting these results and questions at a conference when Wis Comfort asked whether one shouldn't be able to do the same thing with $p\in\{r^n\colon n\in\mathbb{N}\}^*$ for any prime r. The fact that he was right is part of the content of Section 2 where we investigate groups generated by $q+p+q$, where q is a minimal idempotent and p is an element of $\{r^n\colon n\in\mathbb{N}\}^*$ for some $r\in\mathbb{N}\setminus\{1\}$. We express here our gratitude to Prof. Comfort for raising the issue as well as much helpful correspondence on the subject.

In Section 3 we investigate groups generated by $q+p+q$, where q is a minimal idempotent and p is an element of $\mathbb{N}^*$ such that $\{n\in\mathbb{N}\colon n\mathbb{N}\in p\}$ is infinite. We do not know of any minimal idempotent q with the property that there exists $p\in\mathbb{N}^*$ such that $\{n\in\mathbb{N}\colon n\mathbb{N}\in p\}$ is infinite and the group generated by $q+p+q$ is not discrete. (And on the other hand, we do not know whether every minimal idempotent has this property.) We show in Section 4 that if a minimal idempotent has the property that there exists $p\in\mathbb{N}^*$ such that $\{n\in\mathbb{N}\colon n\mathbb{N}\in p\}$ is infinite and the group generated by $q+p+q$ is not discrete, then every idempotent in the same minimal right ideal has that property and there are $\mathfrak{c}$ different minimal right ideals whose idempotents have that property.

We close this introduction with some notation and facts which we will be using throughout.

Definition 1.2. If $A\subseteq\mathbb{N}$, $A^*=cl_{\beta\mathbb{N}}(A)\setminus A$.

Definition 1.3. If X is discrete, Y is a completely regular Hausdorff space, and $f:X\to Y$, then $\tilde{f}:\beta X\to\beta Y$ will denote the continuous extension of f.

In particular, if Y is compact, then $\tilde{f}:\beta X\to Y$.

Note that it certainly is possible that the group generated by $q+p+q$ is finite, in which case by Zelenyuk's Theorem [6] (see [3, Section 7.1]) that group is $\{q\}$. Specifically, if p is any nonminimal idempotent, then by [3, Theorem 1.60], there is a minimal idempotent $q\leqslant p$, and then $q+p+q=q$. We note that given any minimal idempotent q, one may show by a modification of the proof of [3, Theorem 8.22] that there exists $p\in(c\ell K(\beta\mathbb{N}))\setminus(\mathbb{N}^*+\mathbb{N}^*)$ such that $p+q=q$. In particular, such p is not an idempotent.

Definition 1.4. Let $r\in\mathbb{N}\setminus\{1\}$. Then $\mathbb{H}_r=\bigcap_{n=1}^{\infty}c\ell(r^n\mathbb{N})$.

Notice that if q is a minimal idempotent in $\beta\mathbb{N}$, then for all $p\in\beta\mathbb{N}$, $q+p+q\in H(q)$. Also, by [3, Lemma 6.6], all idempotents of $\beta\mathbb{N}$ are in $\mathbb{H}_r$ for each $r\in\mathbb{N}\setminus\{1\}$.

Definition 1.5. Let $q\in E(\beta\mathbb{N})$ and let $p\in\beta\mathbb{N}$ such that $q+p+q\in H(q)$.

(a) $\psi_{q,p}(0)=q$ and for $n\in\mathbb{N}$, $\psi_{q,p}(n)$ is the sum of $q+p+q$ with itself n times and $\psi_{q,p}(-n)$ is the inverse of $\psi_{q,p}(n)$ in the group $H(q)$.
(b) $\Psi_{q,p}=\{\psi_{q,p}(n)\colon n\in\mathbb{Z}\}$.

We refer to $\Psi_{q,p}$ as *the group generated by* $q+p+q$.

Lemma 1.6. *Let $q\in E(\beta\mathbb{N})$ and let $p\in\beta\mathbb{N}$ such that $q+p+q\in H(q)$. Then $\Psi_{q,p}$ is discrete if and only if $q\notin c\ell\{\psi_{q,p}(n)\colon n\in\mathbb{Z}\setminus\{0\}\}$.*

Proof. The necessity is trivial. For the sufficiency, if $m\in\mathbb{N}$ and

$$\psi_{q,p}(m)\in c\ell\{\psi_{q,p}(n)\colon n\in\mathbb{Z}\setminus\{m\}\},$$

apply $\rho_{\psi_{q,p}(-m)}$ to conclude that $q\in c\ell\{\psi_{q,p}(n)\colon n\in\mathbb{Z}\setminus\{0\}\}$. □

2. Groups generated by elements living on powers

As we have remarked, given a minimal idempotent q, we do not know whether there is any $p \in \{2^n\colon n \in \mathbb{N}\}^*$ such that $\Psi_{q,p}$ is not both infinite and discrete. (It will always be infinite.) Similarly, for any $r \in \mathbb{N} \setminus \{1\}$ we do not know whether there is any $p \in \{r^n\colon n \in \mathbb{N}\}^*$ such that $\Psi_{q,p}$ is not both infinite and discrete. We show in Corollary 2.17 that, given q, there is at most one prime r such that there exists $p \in \{r^n\colon n \in \mathbb{N}\}^*$ such that $\Psi_{q,p}$ is not both infinite and discrete.

We first fill in a glaring omission from [3]. In [3, Theorem 2.11(b)] we presented a proof that maximal groups in the same minimal right ideal of a compact Hausdorff right topological semigroup are topologically and algebraically isomorphic. But we neglected to note the simple form which the combined isomorphism and homeomorphism took. (We could appeal to our proof from [3], noting that the map produced there is really ρ_f. But the entire proof is so simple that we present it below.)

Definition 2.1. Let S be a semigroup. If $q_1, q_2 \in E(S)$, we write $q_1 \sim_R q_2$ if and only if $q_1 q_2 = q_2$ and $q_2 q_1 = q_1$.

We observe that $\sim_R$ is an equivalence relation and that, for $q_1, q_2 \in E(S)$, $q_1 \sim_R q_2$ if and only if $q_1 S = q_2 S$. If S has a minimal right ideal R, $q_1 \in E(R)$, and $q_2 \in E(S)$, then $q_1 \sim_R q_2$ if and only if $q_2 \in R$.

Lemma 2.2. *Let $(S,\cdot)$ be a Hausdorff right topological semigroup and let $e, f \in E(S)$ satisfy $e \sim_R f$. Then the restriction of ρ_f to eSe is an isomorphism and a homeomorphism from eSe onto fSf. Furthermore, if $s \in S$ and $ese \in H(e)$, then $esf \in H(f)$ and the groups generated by ese and $fesf$ are topologically isomorphic.*

Proof. First, if $x \in eSe$, then $\rho_f(x) = xf = exf = fexf = fxf$ and $\rho_e(\rho_f(x)) = xfe = xe = x$. If $z \in fSf$, then $\rho_f(eze) = efzef = fzf = z$. Since ρ_f and ρ_e are continuous, we have that the restriction of ρ_f to eSe is a homeomorphism from eSe onto fSf. To see that it is a homomorphism, let $x, y \in eSe$. Then $\rho_f(xy) = xeyf = xfeyf = xfyf = \rho_f(x)\rho_f(y)$.

Now suppose that $s \in S$ and that $ese \in H(e)$. Since $\rho_f(ese) = esf = fesf$, $esf \in H(f)$ and ρ_f defines an isomorphism and a homeomorphism from the group generated by ese onto the group generated by $fesf$. $\square$

The following well-known lemma will be our main tool in finding discrete groups in $\beta\mathbb{N}$.

Lemma 2.3. *Let A and B be countable subsets of $\beta\mathbb{N}$ for which $A \cap \overline{B} = \overline{A} \cap B = \emptyset$. Then $\overline{A} \cap \overline{B} = \emptyset$.*

Proof. [3, Theorem 3.40]. $\square$

Several of the following results have as part of their hypotheses that $q_1 \sim_R q_2$. We have noted that this holds if q_1 and q_2 are members of the same minimal right ideal of $\beta\mathbb{N}$. It also holds if q_1 and q_2 are members of the same minimal right ideal of a compact subsemigroup of $\beta\mathbb{N}$. It can hold for elements that are far from the smallest ideal of $\beta\mathbb{N}$. If $f : \mathbb{N} \to \beta\mathbb{N}$ is a homomorphism, then its continuous extension $\tilde{f} : \beta\mathbb{N} \to \beta\mathbb{N}$ is also a homomorphism. (This result is due to P. Milnes [5]. See [3, Corollary 4.21].) In this case, if $q_1 \sim_R q_2$, then $\tilde{f}(q_1) \sim_R \tilde{f}(q_2)$.

We now introduce some notation based on the base r expansion of integers.

Definition 2.4. Let $r \in \mathbb{N} \setminus \{1\}$ and let $x \in \mathbb{N}$.

(1) $\kappa_{r,x}$ is the unique function from ω to $\{0, 1, \ldots, r-1\}$ such that $x = \sum_{t=0}^{\infty} \kappa_{r,x}(t) r^t$.
(2) $\operatorname{supp}_r(x) = \{t \in \omega\colon \kappa_{r,x}(t) \neq 0\}$.

The function $\xi_{A,n}$ which we define next depends also on r, but we suppress that dependence since we will only be using one r at a time.

Definition 2.5. For $n \in \mathbb{N}$ and $A \subseteq \mathbb{N}$, we define $\xi_{A,n} : \mathbb{N} \to \mathbb{Z}_n$ by $\xi_{A,n}(x) \equiv |\operatorname{supp}_r(x) \cap A| \pmod{n}$.

Lemma 2.6. *Let $r \in \mathbb{N} \setminus \{1\}$ and let $A \subseteq \mathbb{N}$. Then $\tilde{\xi}_{A,n}$ is a homomorphism on $\mathbb{H}_r$ and so $\tilde{\xi}_{A,n}(q) = 0$ for every $q \in E(\beta\mathbb{N})$. Let $q \in E(\beta\mathbb{N})$ and let $p \in \{r^n\colon n \in \mathbb{N}\}^*$ such that $q + p + q \in H(q)$.*

(a) *If $\{r^t \in \mathbb{N}\colon t \in A\} \in p$, then $\tilde{\xi}_{A,n}(\psi_{q,p}(k)) \equiv k \pmod{n}$ for every $k \in \mathbb{Z}$.*
(b) *If $\{r^t \in \mathbb{N}\colon t \in A\} \in p$ and $q' \in E(\beta\mathbb{N})$ such that $q \sim_R q'$, then $q' + p + q \in H(q)$ and $\tilde{\xi}_{A,n}(\psi_{q,q'+p}(k)) \equiv k \pmod{n}$ for every $k \in \mathbb{Z}$.*
(c) *If $\{r^t \in \mathbb{N}\colon t \in A\} \notin p$, then $\tilde{\xi}_{A,n}(\psi_{q,p}(k)) = 0$ for every $k \in \mathbb{Z}$.*
(d) *If $\{r^t \in \mathbb{N}\colon t \in A\} \notin p$ and $q' \in E(\beta\mathbb{N})$ such that $q \sim_R q'$, then $q' + p + q \in H(q)$ and $\tilde{\xi}_{A,n}(\psi_{q,q'+p}(k)) = 0$ for every $k \in \mathbb{Z}$.*

Proof. Since $\xi_{A,n}(x_1+x_2)=\xi_{A,n}(x_1)+\xi_{A,n}(x_2)$ whenever x_1 and x_2 in $\mathbb{N}$ satisfy $\max(\mathrm{supp}_r(x_1))<\min(\mathrm{supp}_r(x_2))$, it follows that $\tilde{\xi}_{A,n}$ is a homomorphism on $\mathbb{H}_r$ by [3, Theorem 4.21]. Hence, if $q\in E(\beta\mathbb{N})$, $\tilde{\xi}_{A,n}(q)=0$.

(a) Since $\xi_{A,n}(r^t)=1$ for $t\in A$ and $\{r^t\colon t\in A\}\in p$, we have that $\tilde{\xi}_{A,n}(p)=1$. So $\tilde{\xi}_{A,n}(\psi_{q,p}(1))=1$ and it follows that $\tilde{\xi}_{A,n}(\psi_{q,p}(k))\equiv k \pmod n$ for every $k\in\mathbb{Z}$.

(b) By Lemma 2.2, $q'+p+q\in H(q)$. Since q, q', and p are in $\mathbb{H}_r$, we have that $\tilde{\xi}_{A,n}(\psi_{q,q'+p}(1))=1$ and it follows that $\tilde{\xi}_{A,n}(\psi_{q,q'+p}(k))\equiv k \pmod n$ for every $k\in\mathbb{Z}$.

Since $\xi_{A,n}(r^t)=0$ for $t\notin A$, (c) and (d) follow in a nearly identical fashion. □

Lemma 2.7. *Let $r\in\mathbb{N}\setminus\{1\}$, let $q\in E(\beta\mathbb{N})$ and let $p\in\{r^n\colon n\in\mathbb{N}\}^*$ such that $q+p+q\in H(q)$. Then $\Psi_{q,p}$ is infinite.*

Proof. This is an immediate consequence of Lemma 2.6(a). □

We now extend Theorem 1.1 (in Theorem 2.9). (The construction is essentially the same as that of [4, Theorem 2.1].)

Lemma 2.8. *Let $r\in\mathbb{N}\setminus\{1\}$ and let $q_1,q_2\in E(\beta\mathbb{N})$ satisfy $q_1\sim_R q_2$. Let p_1 and p_2 be distinct elements of $\{r^n\colon n\in\mathbb{N}\}^*$ such that $q_i+p_j+q_i\in H(q_i)$ for $i,j\in\{1,2\}$. If Ψ_{q_1,p_1} is not discrete, then Ψ_{q_2,p_2} is discrete.*

Proof. Pick disjoint subsets A_1 and A_2 of $\mathbb{N}$ such that $\{r^t\colon t\in A_i\}\in p_i$ for $i\in\{1,2\}$. Assume that Ψ_{q_1,p_1} is not discrete. Then by Lemma 2.2, Ψ_{q_2,q_1+p_1} is also not discrete and so $q_2\in c\ell\{\psi_{q_2,q_1+p_1}(m)\colon m\in\mathbb{Z}\setminus\{0\}\}$. We claim that

$$q_2\notin c\ell\bigl\{\psi_{q_2,p_2}(m)\colon m\in\mathbb{Z}\setminus\{0\}\bigr\}.$$

To see this, it suffices by Lemma 2.3 to show that

$$\bigl\{\psi_{q_2,q_1+p_1}(m)\colon m\in\mathbb{Z}\setminus\{0\}\bigr\}\cap c\ell\bigl\{\psi_{q_2,p_2}(m)\colon m\in\mathbb{Z}\setminus\{0\}\bigr\}=\emptyset \quad\text{and}$$
$$\bigl\{\psi_{q_2,p_2}(m)\colon m\in\mathbb{Z}\setminus\{0\}\bigr\}\cap c\ell\bigl\{\psi_{q_2,q_1+p_1}(m)\colon m\in\mathbb{Z}\setminus\{0\}\bigr\}=\emptyset.$$

Let $k\in\mathbb{Z}\setminus\{0\}$ and let $n=|k|+1$. By Lemma 2.6(b),

$$\bigl\{u\in\beta\mathbb{N}\colon \tilde{\xi}_{A_1,n}(u)\equiv k \pmod n\bigr\}$$

is a neighborhood of $\psi_{q_2,q_1+p_1}(k)$ while by Lemma 2.6(c), $\tilde{\xi}_{A_1,n}(\psi_{q_2,p_2}(m))=0$ for all $m\in\mathbb{Z}$. Similarly, by Lemma 2.6(a), $\{u\in\beta\mathbb{N}\colon \tilde{\xi}_{A_2,n}(u)\equiv k \pmod n\}$ is a neighborhood of $\psi_{q_2,p_2}(k)$ while by Lemma 2.6(d), $\tilde{\xi}_{A_2,n}(\psi_{q_2,q_1+p_1}(m))=0$ for all $m\in\mathbb{Z}$. □

Theorem 2.9. *Let $r\in\mathbb{N}\setminus\{1\}$ and let $q_1,q_2\in E(\beta\mathbb{N})$ satisfy $q_1\sim_R q_2$. Let p_1 and p_2 be distinct elements of $\{r^n\colon n\in\mathbb{N}\}^*$ such that $q_i+p_j+q_i\in H(q_i)$ for $i,j\in\{1,2\}$. Then Ψ_{q_1,p_1} and Ψ_{q_2,p_1} are both infinite and discrete or Ψ_{q_1,p_2} and Ψ_{q_2,p_2} are both infinite and discrete.*

Proof. By Lemma 2.7 each of these groups is infinite. Assume that Ψ_{q_1,p_1} is not discrete. By Lemma 2.8, Ψ_{q_2,p_2} is discrete while by Lemma 2.8 with $q_2=q_1$, Ψ_{q_1,p_2} is discrete. □

Corollary 2.10. *Let R be a minimal right ideal in $\beta\mathbb{N}$ and let p_1 and p_2 be distinct elements of $\{r^n\colon n\in\mathbb{N}\}^*$. Then Ψ_{q,p_1} is infinite and discrete for every $q\in E(R)$ or Ψ_{q,p_2} is infinite and discrete for every $q\in E(R)$.*

We now show that there is a sense, to be made precise in Corollary 2.14, in which for each $p\in\{r^n\colon n\in\mathbb{N}\}^*$, most idempotents $q\in K(\beta\mathbb{N})$ have the property that $\Psi_{q,p}$ is infinite and discrete.

Again, since we only deal with one r at a time, we suppress the dependence of the function l_A on r in the next definition.

Definition 2.11. Let $r\in\mathbb{N}\setminus\{1\}$ and let $A\subseteq\mathbb{N}$. Then for $x\in\mathbb{N}$,

$$l_A(x)=\begin{cases}\max(\mathrm{supp}_r(x)\setminus A) & \text{if } \mathrm{supp}_r(x)\setminus A\neq\emptyset,\\ 0 & \text{if } \mathrm{supp}_r(x)\subseteq A.\end{cases}$$

Theorem 2.12. *Let $r\in\mathbb{N}\setminus\{1\}$ and let $p\in\{r^n\colon n\in\mathbb{N}\}^*$. Let $q_1,q_2\in E(\beta\mathbb{N})$ satisfy $q_1\sim_R q_2$ and $q_i+p+q_i\in H(q_i)$ for $i\in\{1,2\}$. Assume that there exists $A\subseteq\mathbb{N}$ such that $\mathbb{N}\setminus A$ is infinite, $\{r^n\colon n\in A\}\in p$, $\tilde{l}_A(q_1)\neq\tilde{l}_A(q_2)$, and $\{x\in\mathbb{N}\colon \mathrm{supp}_r(x)\subseteq A\}\notin q_1\cup q_2$. Then $\Psi_{q_1,p}$ is infinite and discrete or $\Psi_{q_2,p}$ is infinite and discrete.*

Proof. Pick A as guaranteed and let $D = \{x \in \mathbb{N}\colon\ \mathrm{supp}_r(x) \setminus A \neq \emptyset\}$. Since $\tilde{l}_A(q_1) \neq \tilde{l}_A(q_2)$, and $\{x \in \mathbb{N}\colon\ \mathrm{supp}_r(x) \subseteq A\} \notin q_1 \cup q_2$, we may pick B_1 and B_2 such that $\{A, B_1, B_2\}$ is a partition of ω and for $i \in \{1,2\}$, $\{x \in \mathbb{N}\colon\ \max(\mathrm{supp}_r(x) \setminus A) \in B_i\} \in q_i$.

By Lemma 2.7, $\Psi_{q_1,p}$ and $\Psi_{q_2,p}$ are infinite. Assume that $\Psi_{q_1,p}$ is not discrete. Then by Lemma 2.2, Ψ_{q_2,q_1+p} is also not discrete and so

$$q_2 \in c\ell\bigl\{\psi_{q_2,q_1+p}(m)\colon\ m \in \mathbb{Z} \setminus \{0\}\bigr\}.$$

We claim that $q_2 \notin c\ell\{\psi_{q_2,p}(m)\colon\ m \in \mathbb{Z} \setminus \{0\}\}$. To see this, it suffices by Lemma 2.3 to show that

$$\bigl\{\psi_{q_2,q_1+p}(m)\colon\ m \in \mathbb{Z} \setminus \{0\}\bigr\} \cap c\ell\bigl\{\psi_{q_2,p}(m)\colon\ m \in \mathbb{Z} \setminus \{0\}\bigr\} = \emptyset \quad \text{and}$$
$$\bigl\{\psi_{q_2,p}(m)\colon\ m \in \mathbb{Z} \setminus \{0\}\bigr\} \cap c\ell\bigl\{\psi_{q_2,q_1+p}(m)\colon\ m \in \mathbb{Z} \setminus \{0\}\bigr\} = \emptyset.$$

To this end, let $k \in \mathbb{Z} \setminus \{0\}$. We shall show that there is a neighborhood of $\psi_{q_2,q_1+p}(k)$ which misses $\{\psi_{q_2,p}(m)\colon\ m \in \mathbb{Z} \setminus \{0\}\}$ and there is a neighborhood of $\psi_{q_2,p}(k)$ which misses $\{\psi_{q_2,q_1+p}(m)\colon\ m \in \mathbb{Z} \setminus \{0\}\}$.

Let $n = |k| + 1$ (or any other integer bigger than $|k|$). For $x \in D$, define $b_1(x)$, $b_2(x)$, and $a(x)$ in $\mathbb{Z}_n$ as follows.

(1) $a(x) \equiv |\{t \in \mathrm{supp}_r(x)\colon\ t < \min(\mathrm{supp}_r(x) \setminus A)\}| \pmod n$.
(2) For $i \in \{1,2\}$, if $\mathrm{supp}_r(x) \cap B_i = \emptyset$, then $b_i(x) = 0$. Otherwise, $b_i(x) \equiv |\{t \in \mathrm{supp}_r(x)\colon\ t > \max(\mathrm{supp}_r(x) \cap B_i)\}| \pmod n$.

Let $\alpha = \tilde{a}(q_1)$. Since $q_2 \in \mathbb{H}_r$, we have that $\tilde{a} \circ \rho_{q_2}$ agrees with $\tilde{a}$ on D, and so $\alpha = \tilde{a}(q_1 + q_2)$. Since $q_1 \sim_R q_2$, $q_1 + q_2 = q_2$ and so $\alpha = \tilde{a}(q_2)$. For $i \in \{1,2\}$ let $\gamma_i = \tilde{b}_i(q_i)$.

Given $x \in D$ and $s, v \in \mathrm{supp}_r(x)$ with $s < v$, let

$$C(s,v,x) = \bigl\{t \in \mathrm{supp}_r(x)\colon\ s < t < v\bigr\}.$$

For $i \in \{1,2\}$ define $\theta_i(x) \in \mathbb{Z}_n$ by

$$\theta_i(x) \equiv \Bigl|\Bigl\{x \in \mathrm{supp}_r(x) \cap B_i\colon\ \bigl(\exists v \in \mathrm{supp}_r(x) \setminus A\bigr)\ \bigl(C(s,v,x) \subseteq A \text{ and}$$
$$\bigl|C(s,v,x)\bigr| \equiv \alpha + \gamma_i + 1 \ (\mathrm{mod}\ n)\bigr)\Bigr\}\Bigr| \pmod n.$$

As in the proof of Lemma 2.6, one shows that each $\tilde{\theta}_i$ is a homomorphism on $\mathbb{H}_r$ and that for each $m \in \mathbb{Z}$,

(a) $\tilde{\theta}_1(\psi_{q_2,q_1+p}(m)) \equiv m \pmod n$;
(b) $\tilde{\theta}_1(\psi_{q_2,p}(m)) = 0$;
(c) $\tilde{\theta}_2(\psi_{q_2,p}(m)) \equiv m \pmod n$; and
(d) $\tilde{\theta}_2(\psi_{q_2,q_1+p}(m)) = 0$.

Therefore, $c\ell\{x \in D\colon\ \tilde{\theta}_1(x) \equiv k \pmod n\}$ is a neighborhood of $\psi_{q_2,q_1+p}(k)$ which misses $\{\psi_{q_2,p}(m)\colon\ m \in \mathbb{Z} \setminus \{0\}\}$ and $c\ell\{x \in D\colon\ \tilde{\theta}_2(x) \equiv k \pmod n\}$ is a neighborhood of $\psi_{q_2,p}(k)$ which misses $\{\psi_{q_2,q_1+p}(m)\colon\ m \in \mathbb{Z} \setminus \{0\}\}$. □

We observe that the assumption that $\{x \in \mathbb{N}\colon\ \mathrm{supp}_r(x) \subseteq A\} \notin q_1 \cup q_2$ in Theorem 2.12 holds automatically if q_1 and q_2 are in $K(\beta\mathbb{N})$. This is a well-known fact, but we have not been able to find a reference for it. (In the case $r = 2$, it follows from [1, Theorem 2.10].) We present the simple proof now. Given a set X, $\mathcal{P}_f(X)$ is the set of finite nonempty subsets of X.

Lemma 2.13. *Let $r \in \mathbb{N} \setminus \{1\}$, let $A \subseteq \mathbb{N}$ such that $\mathbb{N} \setminus A$ is infinite, and let*

$$B = \bigl\{x \in \mathbb{N}\colon\ \mathrm{supp}_r(x) \subseteq A\bigr\}.$$

Then $\overline{B} \cap K(\beta\mathbb{N}) = \emptyset$.

Proof. By [3, Theorem 4.40] we need to show that B is not piecewise syndetic. That is, $(\forall G \in \mathcal{P}_f(\mathbb{N}))$ $(\exists F \in \mathcal{P}_f(\mathbb{N}))$ $(\forall x \in \mathbb{N})$ $(\exists y \in F)$ $(\forall t \in G)$ $(t + y + x \notin B)$. So let $G \in \mathcal{P}_f(\mathbb{N})$ be given and pick $k \in \mathbb{N} \setminus A$ such that $r^k > \max G$. Let $F = \{1, 2, \ldots, r^{k+1}\}$. Let $x \in \mathbb{N}$ be given and pick $y \in F$ such that r^k divides $y + x$ and r^{k+1} does not divide $y + x$. Then $k = \min \mathrm{supp}_r(y + x)$ so for all $t \in G$, $k \in \mathrm{supp}_r(t + y + x)$. □

Given $r \in \mathbb{N} \setminus \{1\}$ and $A \subseteq \mathbb{N}$ such that $\mathbb{N} \setminus A$ is infinite and given any $v \in (\mathbb{N} \setminus A)^*$ (of which there are $2^{\mathfrak{c}}$), it is routine to show that $L_v = \{q \in \mathbb{H}_r\colon\ \tilde{l}_A(q) = v\}$ is a left ideal of $\mathbb{H}_r$. Thus, if R is a minimal right ideal of $\beta\mathbb{N}$, then $\{L_v \cap R\colon\ v \in (\mathbb{N} \setminus A)^*\}$ is a partition of $R \cap \mathbb{H}_r$, each cell of which has an idempotent. The following corollary says that, given $p \in \{r^n\colon\ n \in A\}^*$, at most one cell of this partition includes an idempotent q for which $\Psi_{q,p}$ is not both infinite and discrete.

Corollary 2.14. *Let $r \in \mathbb{N} \setminus \{1\}$, let $p \in \{r^n\colon n \in \mathbb{N}\}^*$ and let R be a minimal right ideal of $\beta\mathbb{N}$. Then for any $A \subseteq \mathbb{N}$ such that $\mathbb{N} \setminus A$ is infinite and $p \in \{r^n\colon n \in A\}^*$, there exists $v \in (\mathbb{N} \setminus A)^*$ such that $\Psi_{q,p}$ is infinite and discrete for every $q \in E(R)$ for which $\tilde{l}_A(q) \neq v$.*

Proof. Let $A \subseteq \mathbb{N}$ such that $\mathbb{N} \setminus A$ is infinite and $p \in \{r^n\colon n \in A\}^*$. Assume there is some $q_1 \in E(R)$ such that $\Psi_{q_1,p}$ is not discrete. Let $v = \tilde{l}_A(q_1)$. Given any $q_2 \in E(R)$ such that $\tilde{l}_A(q_2) \neq v$, we have by Lemma 2.13 that $\{x \in \mathbb{N}\colon \operatorname{supp}_r(x) \subseteq A\} \notin q_1 \cup q_2$, so by Theorem 2.12, $\Psi_{q_2,p}$ is infinite and discrete. □

Lemma 2.15. *Let $r, s \in \mathbb{N} \setminus \{1\}$ and assume that there is a prime factor α of s which does not divide r. Let $p \in \{r^n\colon n \in \mathbb{N}\}^*$ and let $q \in E(K(\beta\mathbb{N}))$.*

Then $\Psi_{q,p} \cap \mathbb{H}_s = \{q\}$.

Proof. For $k \in \mathbb{N} \setminus \{1\}$, define $h_k : \mathbb{N} \to \{0, 1, \ldots, k-1\}$ by $h_k(n) \equiv n \pmod k$. Let $\tilde{h}_k : \beta\mathbb{N} \to \{0, 1, \ldots, k-1\}$ be its continuous extension. Note that if $v \in \mathbb{N}^*$, $k \in \mathbb{N} \setminus \{1\}$, and $i \in \{0, 1, \ldots, k-1\}$, then $\tilde{h}_k(v) = i$ if and only if $k\mathbb{N} + i \in v$. Also $\tilde{h}_k$ is a homomorphism onto $\mathbb{Z}_k$ by [3, Corollary 4.22].

We show first that if $n \in \mathbb{N}$, then $\psi_{q,p}(n) \notin \mathbb{H}_s$. So let $n \in \mathbb{N}$ and pick $k \in \mathbb{N}$ such that $\alpha^k > n$. Let $i = \tilde{h}_{s^k}(p)$. Then $s^k\mathbb{N} + i \in p$ so pick m such that $r^m \in s^k\mathbb{N} + i$. Then s^k divides $r^m - i$. If α divided i, then α would divide r^m and thus would divide r, so α does not divide i. Since q is an idempotent and $\tilde{h}_{s^k}$ is a homomorphism we have that $\tilde{h}_{s^k}(q) = 0$ and therefore $\tilde{h}_{s^k}(\psi_{q,p}(n)) \equiv ni \pmod{s^k}$. If we had $\psi_{q,p}(n) \in \overline{s^k\mathbb{N}}$, we would have $ni \equiv 0 \pmod{s^k}$ and so would have α^k divides ni. Since α does not divide i, this would imply that α^k divides n, while $n < \alpha^k$.

Finally, $\tilde{h}_{s^k}(\psi_{q,p}(-n)) + \tilde{h}_{s^k}(\psi_{q,p}(n)) \equiv 0 \pmod{s^k}$ so $\tilde{h}_{s^k}(\psi_{q,p}(-n)) \neq 0$. □

Theorem 2.16. *Let $r, s \in \mathbb{N} \setminus \{1\}$ and assume that each has a prime factor which does not divide the other. Let R be a minimal right ideal in $\beta\mathbb{N}$. Either for all $p \in \{r^n\colon n \in \mathbb{N}\}^*$, $\Psi_{q,p}$ is infinite and discrete for every $q \in E(R)$, or for all $u \in \{s^n\colon n \in \mathbb{N}\}^*$, $\Psi_{q,u}$ is infinite and discrete for every $q \in E(R)$.*

Proof. By Lemma 2.7 we have that for all $p \in \{r^n\colon n \in \mathbb{N}\}^*$, $\Psi_{q,p}$ is infinite, and for all $u \in \{s^n\colon n \in \mathbb{N}\}^*$, $\Psi_{q,u}$ is infinite.

Suppose we have $p \in \{r^n\colon n \in \mathbb{N}\}^*$ and $q \in E(R)$ such that $\Psi_{q,p}$ is not discrete. Let $v \in E(R)$ and $u \in \{s^n\colon n \in \mathbb{N}\}^*$. We shall show that $\Psi_{v,u}$ is discrete. By Lemma 2.2, $\Psi_{v,q+p}$ is not discrete and so $q \in cl(\{\psi_{v,q+p}(m)\colon m \in \mathbb{Z} \setminus \{0\}\})$.

Let $k \in \mathbb{Z} \setminus \{0\}$. Then, by Lemma 2.15, $\psi_{v,u}(k) \notin cl(\Psi_{v,q+p})$ because $\Psi_{v,q+p} \subseteq \mathbb{H}_r$. Similarly, $\psi_{v,q+p}(k) \notin cl(\Psi_{v,u})$ because $\Psi_{v,u} \subseteq \mathbb{H}_s$. By Lemma 2.3, $cl(\{\psi_{v,u}(k)\colon k \in \mathbb{Z} \setminus \{0\}\})$ does not meet $c\ell(\{\psi_{v,q+p}(k)\colon k \in \mathbb{Z} \setminus \{0\}\})$. So $q \notin cl(\{\psi_{v,u}(k)\colon k \in \mathbb{Z} \setminus \{0\}\})$ and $\Psi_{v,u}$ is discrete. □

Corollary 2.17. *Let $q \in E(K(\beta\mathbb{N}))$. There is at most one prime r such that there exists $p \in \{r^n\colon n \in \mathbb{N}\}^*$ such that $\Psi_{q,p}$ is not discrete, and there is at most one such p.*

Proof. This is an immediate consequence of Theorems 2.9 and 2.16. □

We have just finished seeing that for any minimal idempotent, there is at most one bad $r \in \mathbb{N} \setminus \{1\}$. We shall show now in Theorem 2.21 that if any r is bad for some minimal idempotent, then all values of r are bad for some minimal idempotent.

Lemma 2.18. *Let $1 < s < r$ in $\mathbb{N}$. Define $\gamma : \{1, 2, \ldots, r-1\} \to \{0, 1, \ldots, s-1\}$ by, for $a \in \{1, 2, \ldots, r-1\}$,*

$$\gamma(a) = \begin{cases} a & \text{if } a < s, \\ 0 & \text{if } a \geqslant s. \end{cases}$$

Define $\varphi : \mathbb{N} \to \omega$ as follows. Let $n \in \mathbb{N}$ and write $n = \sum_{t \in F} a_t r^t$ where $F \in \mathcal{P}_f(\omega)$ and for each $t \in F$, $a_t \in \{1, 2, \ldots, r-1\}$. Then $\varphi(n) = \sum_{t \in F} \gamma(a_t) s^t$. Then the restriction of $\tilde{\varphi}$ to $\mathbb{H}_r$ is a homomorphism and $\tilde{\varphi}[\mathbb{H}_r] = \mathbb{H}_s \cup \{0\}$. If $p \in \{r^n\colon n \in \mathbb{N}\}^*$, then $\tilde{\varphi}(p) \in \{s^n\colon n \in \mathbb{N}\}^*$.

Proof. To see that the restriction of $\tilde{\varphi}$ to $\mathbb{H}_r$ is a homomorphism, we use [3, Theorem 4.21]. Let $x \in \mathbb{N}$ and pick $F \in \mathcal{P}_f(\omega)$ and $\langle a_t \rangle_{t \in F}$ in $\{1, 2, \ldots, r-1\}$ such that $x = \sum_{t \in F} a_t r^t$. Let $k = \max F + 1$ and let $y \in r^k\mathbb{N}$. Pick $G \in \mathcal{P}_f(\omega)$ and $\langle a_t \rangle_{t \in G}$ in $\{1, 2, \ldots, r-1\}$ such that $y = \sum_{t \in G} a_t r^t$. Then $\min G > \max F$ so $x + y = \sum_{t \in F \cup G} a_t r^t$. Thus $\varphi(x + y) = \sum_{t \in F \cup G} \gamma(a_t) r^t = \sum_{t \in F} \gamma(a_t) r^t + \sum_{t \in G} \gamma(a_t) r^t = \varphi(x) + \varphi(y)$.

Notice that for any k, $\varphi((r-1)r^k) = 0$ so we have that $\varphi[r^k\mathbb{N}] = s^k\mathbb{N} \cup \{0\}$. Consequently, $\tilde{\varphi}[\mathbb{H}_r] = \mathbb{H}_s \cup \{0\}$.

Finally, let $p \in \{r^n\colon n \in \mathbb{N}\}^*$. Since $\varphi[\{r^n\colon n \in \mathbb{N}\}] = \{s^n\colon n \in \mathbb{N}\}$ we have $\{s^n\colon n \in \mathbb{N}\} \in \tilde{\varphi}(p)$. Given $n \in \mathbb{N}$, $\varphi^{-1}[\{s^n\}] = \{r^n\}$, so $\tilde{\varphi}(p) \in \{s^n\colon n \in \mathbb{N}\}^*$. □

The proof of the following lemma is an exercise in arithmetic.

Lemma 2.19. *Let $1 < r < s$ in $\mathbb{N}$ and pick the first $k \in \mathbb{N}$ such that $r^k \geqslant s + k$. There is a function $\gamma : \{1, 2, \ldots, r^k - 1\} \xrightarrow{\text{onto}} \{0, 1, \ldots, s - 1\}$ such that $\gamma(r^k - 1) = 0$ and if $t \in \{0, 1, \ldots, k - 1\}$, then $\gamma(r^t) = 1$.*

Lemma 2.20. *Let $1 < r < s$ in $\mathbb{N}$ and pick the first $k \in \mathbb{N}$ such that $r^k \geqslant s + k$. Let γ be chosen as in Lemma 2.19. Define $\varphi : \mathbb{N} \to \omega$ as follows. Let $n \in \mathbb{N}$ and write $n = \sum_{t\in F} a_t r^{kt}$ where $F \in \mathcal{P}_f(\omega)$ and for each $t \in F$, $a_t \in \{1, 2, \ldots, r^k - 1\}$. Then $\varphi(n) = \sum_{t\in F} \gamma(a_t)s^t$. Let $\tilde{\varphi} : \beta\mathbb{N} \to \beta\omega$ be the continuous extension of φ. Then the restriction of $\tilde{\varphi}$ to $\mathbb{H}_r$ is a homomorphism and $\tilde{\varphi}[\mathbb{H}_r] = \mathbb{H}_s \cup \{0\}$. If $p \in \{r^n\colon n \in \mathbb{N}\}^*$, then $\tilde{\varphi}(p) \in \{s^n\colon n \in \mathbb{N}\}^*$.*

Proof. That the restriction of $\tilde{\varphi}$ to $\mathbb{H}_r$ is a homomorphism is established exactly as in the proof of Lemma 2.18.

Now $\mathbb{H}_r = \bigcap_{n=1}^{\infty} \overline{r^{kn}\mathbb{N}}$ so to see that $\tilde{\varphi}[\mathbb{H}_r] = \mathbb{H}_s \cup \{0\}$ it suffices to show that for each $n \in \mathbb{N}$, $\varphi[r^{kn}\mathbb{N}] = s^n\mathbb{N} \cup \{0\}$. Immediately, $\varphi[r^{kn}\mathbb{N}] \subseteq s^n\mathbb{N} \cup \{0\}$ and $\varphi((r^k - 1)r^{nk}) = 0$. Since γ is surjective, we get easily that $s^n\mathbb{N} \subseteq \varphi[r^{kn}\mathbb{N}]$.

Since $\gamma(r^t) = 1$ for each $t \in \{0, 1, \ldots, k - 1\}$, we have that $\varphi[\{r^n\colon n \in \mathbb{N}\}] = \{s^n\colon n \in \mathbb{N}\}$. Further, given $m \in \mathbb{N}$, we have that $\varphi^{-1}[\{s^m\}]$ is finite so $\tilde{\varphi}[\{r^n\colon n \in \mathbb{N}\}^*] = \{s^n\colon n \in \mathbb{N}\}^*$. $\square$

Theorem 2.21. *If there exist $r \in \mathbb{N} \setminus \{1\}$, $p \in \{r^n\colon n \in \mathbb{N}\}^*$, and $q \in E(K(\beta\mathbb{N}))$ such that $\Psi_{q,p}$ is not discrete, then for all $s \in \mathbb{N} \setminus \{1\}$, there exist $p' \in \{s^n\colon n \in \mathbb{N}\}^*$, and $q' \in E(K(\beta\mathbb{N}))$ such that $\Psi_{q',p'}$ is not discrete.*

Proof. Pick r, p, and q as guaranteed by the hypothesis and let $s \in \mathbb{N} \setminus \{1\}$. We may assume $s \neq r$ so by either Lemma 2.18 or Lemma 2.20 we may pick $\varphi : \mathbb{N} \to \omega$ such that the restriction of $\tilde{\varphi}$ to $\mathbb{H}_r$ is a homomorphism, $\tilde{\varphi}[\mathbb{H}_r] = \mathbb{H}_s \cup \{0\}$, and $\tilde{\varphi}(p) \in \{s^n\colon n \in \mathbb{N}\}^*$. Let $q' = \tilde{\varphi}(q)$ and let $p' = \tilde{\varphi}(p)$.

By [3, Theorem 1.65] $K(\mathbb{H}_r) = \mathbb{H}_r \cap K(\beta\mathbb{N})$ so $q \in K(\mathbb{H}_r)$ and thus $q' \in \tilde{\varphi}[K(\mathbb{H}_r)]$. By [3, Exercise 1.7.3 and Theorem 1.65], $\tilde{\varphi}[K(\mathbb{H}_r)] = K(\mathbb{H}_s \cup \{0\}) = K(\mathbb{H}_s) = \mathbb{H}_s \cap K(\beta\mathbb{N})$ so $q' \in K(\beta\mathbb{N})$.

Since $\Psi_{q,p}$ is not discrete, pick $m \in \mathbb{Z}$ such that $\psi_{q,p}(m) \in c\ell\{\psi_{q,p}(n)\colon n \in \mathbb{Z} \setminus \{m\}\}$. Then

$$\begin{aligned}
\psi_{q',p'}(m) &= \tilde{\varphi}\big(\psi_{q,p}(m)\big) \\
&\in \tilde{\varphi}\big[c\ell\big\{\psi_{q,p}(n)\colon n \in \mathbb{Z} \setminus \{m\}\big\}\big] \\
&= c\ell\big\{\tilde{\varphi}\big(\psi_{q,p}(n)\big)\colon n \in \mathbb{Z} \setminus \{m\}\big\} \\
&= c\ell\big\{\psi_{q',p'}(n)\colon n \in \mathbb{Z} \setminus \{m\}\big\}. \qquad \square
\end{aligned}$$

3. Groups generated by elements living on $n\mathbb{N}$ for infinitely many n

Let $\mathbb{I} = \{p \in \beta\mathbb{N}\colon \{n \in \mathbb{N}\colon n\mathbb{N} \in p\}$ is infinite$\}$. All idempotents of $\beta\mathbb{N}$ are in $\mathbb{I}$. Furthermore, members of $\mathbb{I}$ are better understood than the average points of $\beta\mathbb{N}$. For example, one of the major unsolved problems in the algebra of $\beta\mathbb{N}$ is whether there exist any $p, q, r, s \in \mathbb{N}^*$ such that $p + q = r \cdot s$. It is a result discovered independently by several people investigating arithmetic in $\beta\mathbb{N}$ that if $s \in \mathbb{I}$, then there do not exist $p, q, r \in \mathbb{N}^*$ such that $p + q = r \cdot s$.

In this section we show that if $p \in \mathbb{I}$, then there is a minimal idempotent q such that $\Psi_{q,p}$ is infinite and discrete. We then use this result to show that discrete copies of $\mathbb{Z}$ are plentiful in βG for any countable discrete group G.

We use the base 8 expansion of integers. (This is because part of the proof needs four different digits so base 4 is not quite good enough.)

Definition 3.1. For each $x \in \mathbb{N}$, let $\alpha(x) = |\mathrm{supp}_8(x)|$, let $l(x, 1), l(x, 2), \ldots, l(x, \alpha(x))$ enumerate $\mathrm{supp}_8(x)$ in increasing order, and for $t \in \{1, 2, \ldots, \alpha(x)\}$, let $\delta(x, t) = \kappa_{8,x}(l(x, t))$.

Thus, for $x \in \mathbb{N}$, $x = \sum_{t=1}^{\alpha(x)} \delta(x, t)8^{l(x,t)}$.

Definition 3.2. Define the functions $f, g : \mathbb{N} \to \{1, 2, 3, 4, 5, 6, 7\}$ by $f(x) = \delta(x, 1)$ and $g(x) = \delta(x, \alpha(x))$.

Thus $f(x)$ is the least significant nonzero digit and $g(x)$ is the most significant nonzero digit in the base 8 expansion.

Definition 3.3. Assume that distinct a and b in $\{1, 2, 3, 4, 5, 6, 7\}$ have been chosen.

(a) Define the function $\nu_{a,b} : \mathbb{N} \to \omega$ by $\nu_{a,b}(x) = |\{t \in \{1, 2, \ldots, \alpha(x) - 1\}\colon \delta(x, t) = a$ and $\delta(x, t + 1) = b\}|$.
(b) For each $m \in \mathbb{N}$, define the function $\theta_{m,a,b} : \mathbb{N} \to \mathbb{Z}_m$ by $\theta_{m,a,b} \equiv \nu_{a,b} \pmod m$.

Lemma 3.4. *Assume that distinct a and b in $\{1, 2, 3, 4, 5, 6, 7\}$ have been chosen, let $q, r \in \mathbb{H}$, and let $m \in \mathbb{N} \setminus \{1\}$. If $\tilde{g}(q) = a$ and $\tilde{f}(r) = b$, then $\tilde{\theta}_{m,a,b}(q + r) = \tilde{\theta}_{m,a,b}(q) + \tilde{\theta}_{m,a,b}(r) + 1$. In all other cases $\tilde{\theta}_{m,a,b}(q + r) = \tilde{\theta}_{m,a,b}(q) + \tilde{\theta}_{m,a,b}(r)$. Consequently, if q is an idempotent, $\tilde{g}(q) = a$, and $\tilde{f}(q) = b$, then $\tilde{\theta}_{m,a,b}(q) = -1$, and in all other cases $\tilde{\theta}_{m,a,b}(q) = 0$.*

Proof. Let $A = \{x \in \mathbb{N}\colon\ \theta_{m,a,b}(x) = \tilde{\theta}_{m,a,b}(q)$ and $g(x) = \tilde{g}(q)\}$ and let $B = \{x \in \mathbb{N}\colon\ \theta_{m,a,b}(x) = \tilde{\theta}_{m,a,b}(r)$ and $f(x) = \tilde{f}(r)\}$. Then $A \in q$ and $B \in r$.

Assume first that $\tilde{g}(q) = a$ and $\tilde{f}(r) = b$ and let

$$C = \bigl\{x \in \mathbb{N}\colon\ \theta_{m,a,b}(x) = \tilde{\theta}_{m,a,b}(q) + \tilde{\theta}_{m,a,b}(r) + 1\bigr\}.$$

We show that $C \in q + r$ by showing that $A \subseteq \{x \in \mathbb{N}\colon\ -x + C \in r\}$. So let $x \in A$ and let $k = l(x, \alpha(x))$. We claim that $B \cap 8^{k+1}\mathbb{N} \subseteq -x + C$. So let $y \in B \cap 8^{k+1}\mathbb{N}$. Then $g(x) = a$, $f(y) = b$, and $l(y, 1) > l(x, \alpha(x))$ so $\theta_{m,a,b}(x + y) = \theta_{m,a,b}(x) + 1 + \theta_{m,a,b}(y)$ and thus $x + y \in C$.

The proof of each of the other cases is identical except that then $\theta_{m,a,b}(x + y) = \theta_{m,a,b}(x) + \theta_{m,a,b}(y)$. $\square$

Lemma 3.5. *Assume that distinct a and c in $\{1, 2, 3, 4, 5, 6, 7\}$ have been chosen, let $q \in E(K(\beta\mathbb{N}))$, let $p \in \mathbb{H}$, let $m \in \mathbb{N} \setminus \{1\}$, and let $n \in \mathbb{Z}$.*

(a) *If $\tilde{g}(q) = a$, $\tilde{f}(p) = c$, and $\tilde{f}(q) \neq c$, then $\tilde{\theta}_{m,a,c}(\psi_{q,p}(n)) = n \cdot (\tilde{\theta}_{m,a,c}(p) + 1)$.*
(b) *If $\tilde{g}(q) = a$, $\tilde{f}(p) \neq c$, and $\tilde{f}(q) \neq c$, then $\tilde{\theta}_{m,a,c}(\psi_{q,p}(n)) = n \cdot \tilde{\theta}_{m,a,c}(p)$.*
(c) *If $\tilde{g}(q) \neq a$ and $\tilde{f}(q) \neq c$, then $\tilde{\theta}_{m,a,c}(\psi_{q,p}(n)) = n \cdot \tilde{\theta}_{m,a,c}(p)$.*

Proof. In each of these cases, $\Psi_{q,p} \subseteq q + \mathbb{H} + q \subseteq \{r \in \mathbb{H}\colon\ \tilde{f}(r) \neq c\}$ and by Lemma 3.4, $\tilde{\theta}_{m,a,c}$ is a homomorphism on $\{r \in \mathbb{H}\colon\ \tilde{f}(r) \neq c\}$. Thus it suffices to establish these statements for $n \in \omega$. Also, $\psi_{q,p}(0) = q$ and by Lemma 3.4, in each of these cases $\tilde{\theta}_{m,a,c}(q) = 0$, so it suffices to establish the statements for $n \in \mathbb{N}$. Notice that in each case $\tilde{f}(q) \neq c$, so by Lemma 3.4, $\tilde{\theta}_{m,a,c}(p + q) = \tilde{\theta}_{m,a,c}(p) + \tilde{\theta}_{m,a,c}(q) = \tilde{\theta}_{m,a,c}(p)$.

(a) Let $n \in \omega$ and assume that $\tilde{\theta}_{m,a,c}(\psi_{q,p}(n)) = n \cdot (\tilde{\theta}_{m,a,c}(p) + 1)$. Then $\tilde{f}(p + q) = c$ so

$$\begin{aligned}\tilde{\theta}_{m,a,c}\bigl(\psi_{q,p}(n+1)\bigr) &= \tilde{\theta}_{m,a,c}\bigl(\psi_{q,p}(n)\bigr) + \tilde{\theta}_{m,a,c}(q + p + q)\\ &= n \cdot \bigl(\tilde{\theta}_{m,a,c}(p) + 1\bigr) + \tilde{\theta}_{m,a,c}(q) + \tilde{\theta}_{m,a,c}(p + q) + 1\\ &= n \cdot \bigl(\tilde{\theta}_{m,a,c}(p) + 1\bigr) + \tilde{\theta}_{m,a,c}(p) + 1.\end{aligned}$$

(b) This case is identical, except that $\tilde{f}(p + q) \neq c$ so $\tilde{\theta}_{m,a,c}(q + p + q) = \tilde{\theta}_{m,a,c}(q) + \tilde{\theta}_{m,a,c}(p + q)$.
(c) This case is identical, except that $\tilde{g}(q) \neq a$ so $\tilde{\theta}_{m,a,c}(q + p + q) = \tilde{\theta}_{m,a,c}(q) + \tilde{\theta}_{m,a,c}(p + q)$. $\square$

Lemma 3.6. *Assume that $a \in \{1, 2, 3, 4, 5, 6, 7\}$, let $L = \{r \in \mathbb{H}\colon\ \tilde{g}(r) = a\}$, and let $R = \{r \in \mathbb{H}\colon\ \tilde{f}(r) = a\}$. The L is a left ideal of $\mathbb{H}$ and R is a right ideal of $\mathbb{H}$.*

Proof. To see that L is a left ideal of $\mathbb{H}$, let $r \in L$ and $p \in \mathbb{H}$. Let $C = \{x \in \mathbb{N}\colon\ g(x) = a\}$. Then $C \in r$. We claim that for all $x \in \mathbb{N}$, $-x + C \in r$ so that $C \in p + r$. So let $x \in \mathbb{N}$. Then $C \cap 8^{l(x,\alpha(x))+1}\mathbb{N} \subseteq -x + C$.

The proof that R is a right ideal of $\mathbb{H}$ is similar. $\square$

Lemma 3.7. *Assume that $a \in \{1, 2, 3, 4, 5, 6, 7\}$ and let F be a compact subsemigroup of $\mathbb{H}$ such that $F \cap \{r \in \mathbb{H}\colon\ \tilde{f}(r) = \tilde{g}(r) = a\} \neq \emptyset$. Then there exists $q \in E(K(F))$ such that $\tilde{f}(q) = \tilde{g}(q) = a$.*

Proof. Let $L = \{r \in \mathbb{H}\colon\ \tilde{g}(r) = a\}$ and $R = \{r \in \mathbb{H}\colon\ \tilde{f}(r) = a\}$. By Lemma 3.6 $L \cap F$ is a left ideal of F and $R \cap F$ is a right ideal of F so there exist a minimal left ideal L' of F and a minimal right ideal R' of F such that $L' \subseteq L$ and $R' \subseteq R$. By [3, Theorems 1.48 and 1.61], $L' \cap R'$ is a subgroup of F contained in $K(F)$. Let q be the identity of $L' \cap R'$. $\square$

Theorem 3.8. *For every $p \in \mathbb{H}$, there exists $q \in E(K(\beta\mathbb{N}))$ for which $\Psi_{q,p}$ is infinite and discrete.*

Proof. Suppose that $\tilde{f}(p) = c$. We can choose distinct $a, b \in \{1, 2, 3, 4, 5, 6, 7\} \setminus \{c\}$ for which $\tilde{\nu}_{a,c}(p)$ and $\tilde{\nu}_{b,c}(p)$ are both in $\overline{2\mathbb{N}}$ or both in $\overline{2\mathbb{N} + 1}$. Then given any even $m \in \mathbb{N}$, we have that $\tilde{\theta}_{m,a,c}(p) \equiv \tilde{\theta}_{m,b,c}(p) \pmod 2$.

By Lemma 3.7, we can choose $q \in E(K(\beta\mathbb{N}))$ for which $\tilde{f}(q) = \tilde{g}(q) = a$. Let R be the minimal right ideal of $\mathbb{H}$ with $q \in R$ and let $L = \{r \in \mathbb{H}\colon\ \tilde{g}(r) = b\}$. By Lemma 3.6 L is a left ideal of $\mathbb{H}$ so pick a minimal left ideal L' of $\mathbb{H}$ such that $L' \subseteq L$. Let v be the identity of $L' \cap R$. Then $\tilde{g}(v) = b$ and, since $R \subseteq \{r \in \mathbb{H}\colon\ \tilde{f}(r) = a\}$, $\tilde{f}(v) = a$.

Suppose that $\Psi_{q,p}$ is nondiscrete. Then, by Lemma 2.2, $\Psi_{v,q+p}$ is also nondiscrete. We shall show that $\Psi_{v,p}$ must be discrete.

Let $s, t \in \mathbb{Z}$, let $m \in \mathbb{N}$, let $k = \tilde{\theta}_{m,a,c}(p)$, and let $l = \tilde{\theta}_{m,b,c}(p)$. We claim that

(1) $\tilde{\theta}_{m,a,c}(\psi_{v,q+p}(s)) = s(k + 1)$,
(2) $\tilde{\theta}_{m,a,c}(\psi_{v,p}(t)) = tk$,

(3) $\tilde{\theta}_{m,b,c}(\psi_{v,q+p}(s)) = sl$, and
(4) $\tilde{\theta}_{m,b,c}(\psi_{v,p}(t)) = t(l+1)$.

Conclusion (1) follows from Lemma 3.5(c) and the fact from Lemma 3.4 that $\tilde{\theta}_{m,a,c}(q+p) = \tilde{\theta}_{m,a,c}(q) + \tilde{\theta}_{m,a,c}(p) + 1$. Conclusion (2) follows from Lemma 3.5(c). Conclusion (3) follows from Lemma 3.5(b) and the fact from Lemma 3.4 that $\tilde{\theta}_{m,b,c}(q+p) = \tilde{\theta}_{m,b,c}(q) + \tilde{\theta}_{m,b,c}(p)$. Conclusion (4) follows from Lemma 3.5(a).

Since $\Psi_{v,q+p}$ is not discrete, we have by Lemma 1.6 that

$$v \in c\ell\{\psi_{v,q+p}(s)\colon s \in \mathbb{Z}\setminus\{0\}\}.$$

To see that $\Psi_{v,p}$ is discrete, i.e., that $v \notin c\ell\{\psi_{v,p}(t)\colon t \in \mathbb{Z}\setminus\{0\}\}$, it suffices by Lemma 2.3 to show that

$$\{\psi_{v,q+p}(s)\colon s \in \mathbb{Z}\setminus\{0\}\} \cap c\ell\{\psi_{v,p}(t)\colon t \in \mathbb{Z}\setminus\{0\}\} = \emptyset \quad \text{and}$$
$$\{\psi_{v,p}(t)\colon t \in \mathbb{Z}\setminus\{0\}\} \cap c\ell\{\psi_{v,q+p}(s)\colon s \in \mathbb{Z}\setminus\{0\}\} = \emptyset.$$

Suppose first that we have $s \in \mathbb{Z}\setminus\{0\}$ such that $\psi_{v,q+p}(s) \in c\ell\{\psi_{v,p}(t)\colon t \in \mathbb{Z}\setminus\{0\}\}$. Pick $u \in \mathbb{N}$ such that $2^u > s^2$, let $m = 2^u$, let $k = \tilde{\theta}_{m,a,c}(p)$, and let $l = \tilde{\theta}_{m,b,c}(p)$. Recall that $k \equiv l \pmod 2$. Let $W = \{r \in \beta\mathbb{N}\colon \tilde{\theta}_{m,a,c}(r) = s(k+1)$ and $\tilde{\theta}_{m,b,c}(r) = sl\}$. Then W is a neighborhood of $\psi_{v,q+p}(s)$ so pick $t \in \mathbb{Z}\setminus\{0\}$ such that $\psi_{v,p}(t) \in W$.

We then have that the equations $s(k+1) = tk$ and $sl = t(l+1)$ hold in $\mathbb{Z}_m$. Then in $\mathbb{Z}_m$, $stkl = st(k+1)(l+1)$. Since kl and $(k+1)(l+1)$ have opposite parity, 2 does not divide $(k+1)(l+1) - kl$, and so 2^u divides st. That is, in $\mathbb{Z}_m$, $st = 0$. But then in $\mathbb{Z}_m$ we have $s^2(k+1) = stk = 0$ and $s^2 l = st(l+1) = 0$. Since $2^u > s^2$, this says that 2 divides both $k+1$ and l, a contradiction.

In a nearly identical fashion, the assumption that there is some $t \in \mathbb{Z}\setminus\{0\}$ with $\psi_{v,p}(t) \in c\ell\{\psi_{v,q+p}(s)\colon s \in \mathbb{Z}\setminus\{0\}\}$ leads to a contradiction. □

The proof of the following is a straightforward adaptation of the proof of [3, Corollary 7.26].

Lemma 3.9. *Let D be an infinite subset of $\mathbb{N}$. Then $\bigcap_{n\in D}\overline{n\mathbb{N}}$ is topologically and algebraically isomorphic to $\mathbb{H}$.*

Proof. Let $E = \{k \in \mathbb{N}\colon (\exists F \in \mathcal{P}_f(D))\ (k\mathbb{N} = \bigcap_{n\in F} n\mathbb{N})\}$. Then $\bigcap_{n\in D}\overline{n\mathbb{N}} = \bigcap_{n\in E}\overline{n\mathbb{N}}$. Let $\mathcal{B} = \{a + n\mathbb{Z}\colon a \in \mathbb{Z}$ and $n \in E\}$. Note that if $n, m \in E$ and k is the least common multiple of n and m, then $k\mathbb{N} = n\mathbb{N} \cap m\mathbb{N}$, so $k \in E$. Thus if $a, b \in \mathbb{Z}$, $n, m \in \mathbb{N}$, $c \in (a + n\mathbb{Z}) \cap (b + m\mathbb{Z})$, and k is the least common multiple of n and m, then $c \in c + k\mathbb{Z} \subseteq (a + n\mathbb{Z}) \cap (b + m\mathbb{Z})$. Thus $\mathcal{B}$ is a basis for a left invariant topology $\mathcal{T}$ on $\mathbb{Z}$. If $a \in \mathbb{Z}$, $n \in E$, and $c \in \mathbb{Z}\setminus(a + n\mathbb{Z})$, then $(c + n\mathbb{Z}) \cap (a + n\mathbb{Z}) = \emptyset$, so $\mathcal{T}$ is zero-dimensional. Given $a \neq b$ in $\mathbb{Z}$ and $n \in E$ with $n > |a - b|$, $(a + n\mathbb{Z}) \cap (b + n\mathbb{Z}) = \emptyset$, so $\mathcal{T}$ is Hausdorff. Since also no point of ω is isolated with respect to $\mathcal{T}$, by [3, Theorem 7.24 and Lemma 7.25] we have that

$$\bigcap\{cl_{\beta\omega}(W\setminus\{0\})\colon W \text{ is a neighborhood of 0 in } \omega \text{ with respect to } \mathcal{T}\}$$

is topologically and algebraically isomorphic to $\mathbb{H}$. This latter intersection is precisely $\bigcap_{n\in D}\overline{n\mathbb{N}}$. □

Theorem 3.10. *Let $p \in \beta\mathbb{N}$. If $D = \{n \in \mathbb{N}\colon n\mathbb{N} \in p\}$ is infinite, then there exists $q \in E(K(\beta\mathbb{N}))$ such that $\Psi_{q,p}$ is infinite and discrete.*

Proof. Let $T = \bigcap_{n\in D}\overline{n\mathbb{N}}$. By Lemma 3.9, $\bigcap_{n\in D}\overline{n\mathbb{N}}$ is topologically and algebraically isomorphic to $\mathbb{H}$. Let $\mu : \mathbb{H} \to T$ be an isomorphism and a homeomorphism. Let $r = \mu^{-1}(p)$. Pick by Theorem 3.8 an idempotent $s \in K(\mathbb{H})$ such that $\Psi_{s,r}$ is infinite and discrete. By [3, Theorem 1.65], $K(T) = T \cap K(\beta\mathbb{N})$. Therefore $\mu(s) \in K(T) \subseteq K(\beta\mathbb{N})$. And $\Psi_{\mu(s),p}$ is infinite and discrete. □

Of course, if S is a semigroup and $K(\beta S)$ is a proper subset of βS (as is almost always true, as for example if S is cancellative by [3, Theorem 4.36]), then no element of $K(\beta S)$ can be right or left cancelable in βS. On the other hand, by [3, Corollary 8.26], if S is countably infinite and cancellative, then there are right cancelable elements of βS in the closure of the set of minimal idempotents, in particular in the closure of $K(\beta S)$.

Corollary 3.11. *Let $(G, \cdot)$ be a countable discrete group and let $p \in G^*$ be right cancelable in βG.*

(a) *There exists $q \in E(G^*)$ such that $q \cdot p \cdot q$ generates an infinite discrete group.*
(b) *If $p \in c\ell K(\beta G)$, then there exists $q \in E(G^*)$ such that $q \cdot p \cdot q$ generates an infinite discrete group contained in $c\ell K(\beta G)\setminus K(\beta G)$.*

Proof. In the proof of [3, Theorem 8.57] we produced a compact subsemigroup T'_∞ with $p \in T'_\infty$ such that $T'_\infty \cap K(\beta G) = \emptyset$. By [3, Theorem 8.63], T'_∞ is algebraically and topologically isomorphic to $\mathbb{H}$. Thus by Theorem 3.8, there exists $q \in E(K(T'_\infty))$ such that $q \cdot p \cdot q$ generates an infinite discrete group.

Now assume that $p \in c\ell K(\beta G)$. Then $c\ell K(\beta G) \cap T'_\infty \neq \emptyset$. Since $c\ell K(\beta G)$ is an ideal of βG by [3, Theorem 4.44], we have $c\ell K(\beta G) \cap T'_\infty$ is an ideal of T'_∞ and therefore $K(T'_\infty) \subseteq c\ell K(\beta G)$. Therefore $q \in c\ell K(\beta G)$ and so the group generated by $q \cdot p \cdot q$ is contained in $c\ell K(\beta G)$. Since it is contained in T'_∞, it does not meet $K(\beta G)$. $\square$

Corollary 3.12. *Let $(G, \cdot)$ be a countable discrete group. There exist discrete copies of $\mathbb{Z}$ contained in $c\ell K(\beta G) \setminus K(\beta G)$.*

Proof. By [3, Corollary 8.26] there exist right cancelable elements in $c\ell K(\beta G)$ so Corollary 3.11(b) applies. $\square$

4. Bad minimal idempotents

The value judgement in the title of this section refers to minimal idempotents q for which there exists some $p \in \mathbb{H}$ such that $\Psi_{q,p}$ is not discrete. We do not know if any such idempotents exist, but we show here that if they do, they are plentiful. (We suspect that if any minimal idempotents are bad, then they all are, but we can't show that either.)

Recall that $\mathbb{I} = \{p \in \beta\mathbb{N}\colon \{n \in \mathbb{N}\colon n\mathbb{N} \in p\} \text{ is infinite}\}$.

Definition 4.1. (a) $\mathcal{M} = \{q \in E(K(\beta\mathbb{N}))\colon (\exists p \in \mathbb{H})\ (\Psi_{q,p} \text{ is not discrete})\}$.
(b) $\mathcal{M}^* = \{q \in E(K(\beta\mathbb{N}))\colon (\exists p \in \mathbb{I})\ (\Psi_{q,p} \text{ is not discrete})\}$.

Thus $\mathcal{M}$ is the set of bad idempotents. And $\mathcal{M}^*$ is a possibly larger set of candidates for the set of bad idempotents.

Theorem 4.2. *If $\mathcal{M}^* \neq \emptyset$, then $\mathcal{M} \neq \emptyset$.*

Proof. Let $q \in \mathcal{M}^*$ and pick $p \in \mathbb{I}$ such that $\Psi_{q,p}$ is not discrete. Let $D = \{n \in \mathbb{N}\colon n\mathbb{N} \in p\}$. By Lemma 3.9, $T = \bigcap_{n \in D} \overline{n\mathbb{N}}$ is topologically and algebraically isomorphic to $\mathbb{H}$. Let $\mu : T \to \mathbb{H}$ be an isomorphism and a homeomorphism. Then since $K(T) = T \cap K(\beta\mathbb{N})$ and $K(\mathbb{H}) = \mathbb{H} \cap K(\beta\mathbb{N})$, we have that $\mu(q) \in E(K(\beta\mathbb{N}))$. And $\Psi_{\mu(q),\mu(p)}$ is not discrete. So $\mu(q) \in \mathcal{M}$. $\square$

Theorem 4.3. *Let R be a minimal right ideal of $\beta\mathbb{N}$. If $R \cap \mathcal{M}^* \neq \emptyset$, then $E(R) \subseteq \mathcal{M}^*$. If $R \cap \mathcal{M} \neq \emptyset$, then $E(R) \subseteq \mathcal{M}$.*

Proof. Note that if $n \in \mathbb{N}$, $p \in \mathbb{N}^*$, $n\mathbb{N} \in p$, and $q \in E(\beta\mathbb{N})$, then $n\mathbb{N} \in q + p$.

Let $q \in R \cap \mathcal{M}^*$ and pick $p \in \mathbb{I}$ such that $\Psi_{q,p}$ is not discrete. Let $r \in E(R)$. Then by Lemma 2.2 the restriction of ρ_r to $q + \beta\mathbb{N} + q$ is an isomorphism and a homomorphism from $q + \beta\mathbb{N} + q$ onto $r + \beta\mathbb{N} + r$. Therefore, the $\Psi_{r,q+p}$ is not discrete and $q + p \in \mathbb{I}$.

The proof for the second assertion simply replaces $\mathbb{I}$ by $\mathbb{H}$. $\square$

Corollary 4.4. *If $\mathcal{M}^* \neq \emptyset$, then $|\mathcal{M}| = 2^{\mathfrak{c}}$.*

Proof. This is an immediate consequence of Theorem 4.2, Theorem 4.3, and [3, Theorem 6.9]. $\square$

We now proceed to show in Theorem 4.9 that if $\mathcal{M}^* \neq \emptyset$, then there are many minimal right ideals meeting $\mathcal{M}$. We then shall show in Theorem 4.10 that if $p \in \{2^n\colon n \in \mathbb{N}\}^*$ and there exists $q \in E(K(\beta\mathbb{N}))$ such that $\Psi_{q,p}$ is not discrete, then there are many such q for the same p.

We again use base 8 expansions. For each $A \subseteq \omega$ we define $f_A : \mathbb{N} \to \{0, 3, 5\}$ and $g_A : \mathbb{N} \to \{0, 3, 5\}$ as follows. If $\kappa_{8,x}^{-1}[\{3, 5\}] = \emptyset$, then $f_A(x) = g_A(x) = 0$. If $\kappa_{8,x}^{-1}[\{3, 5\}] \neq \emptyset$, then $f_A(x) = \kappa_{8,x}(\min \kappa_{8,x}^{-1}[\{3, 5\}])$ and $g_A(x) = \kappa_{8,x}(\max \kappa_{8,x}^{-1}[\{3, 5\}])$.

We define $h_A : \mathbb{N} \to \mathbb{N}$ to be the mapping which interchanges the digits 3 and 5 in base 8 expansions when these are coefficients of 8^m with $m \in A$. Thus, for each $x \in \mathbb{N}$, $\mathrm{supp}_8(h_A(x)) = \mathrm{supp}_8(x)$ and if $t \in \omega \setminus A$, then $\kappa_{8,h_A(x)}(t) = \kappa_{8,x}(t)$. Also, if $\kappa_{8,x}(t) \in \{0, 1, 2, 4, 6, 7\}$, then $\kappa_{8,h_A(x)}(t) = \kappa_{8,x}(t)$. If $t \in A$ and $\kappa_{8,x}(t) \in \{3, 5\}$, then $\kappa_{8,h_A(x)}(t) = 8 - \kappa_{8,x}(t)$.

Lemma 4.5. *Let $A \subseteq \omega$. The restriction of $\tilde{h}_A$ to $\mathbb{H}$ is an isomorphism and a homeomorphism from $\mathbb{H}$ onto itself. In particular, if $q \in E(K(\beta\mathbb{N}))$, then so is $\tilde{h}_A(q)$.*

Proof. Since h_A is a bijection, $\tilde{h}_A$ is also a bijection by [3, Exercise 3.4.1]. Since $h_A(m + n) = h_A(m) + h_A(n)$ whenever $m, n \in \mathbb{N}$ and $\max \mathrm{supp}_8(m) < \min \mathrm{supp}_8(n)$, $\tilde{h}_A$ is a homomorphism on $\mathbb{H}$ by [3, Lemma 6.3]. If $q \in E(K(\beta\mathbb{N}))$, then $q \in \mathbb{H} \cap K(\beta\mathbb{N}) = k(\mathbb{H})$ so $\tilde{h}_A(q) \in K(\mathbb{H})$. $\square$

Lemma 4.6. *Let $i \in \{1, 2, 3, 4, 5, 6, 7\}$, let A be an infinite subset of ω, and let $B = \{x \in \mathbb{N}\colon \kappa_{8,x}(t) = i \text{ for some } t \in A\}$. Then $K(\mathbb{H}) \subseteq \overline{B}$.*

Proof. It suffices to show that $\overline{B}\cap\mathbb{H}$ is an ideal of $\mathbb{H}$. So let $p\in\overline{B}\cap\mathbb{H}$ and let $q\in\mathbb{H}$. To see that $B\in p+q$ we show that for all $x\in\mathbb{N}$, $-x+B\in p$. So let $x\in\mathbb{N}$ be given. Pick $m\in\mathbb{N}$ such that $8^m>x$. Then $B\cap\mathbb{N}8^m\in p$ and $B\cap\mathbb{N}8^m\subseteq -x+B$. To see that $B\in q+p$, we show that $B\subseteq\{x\in\mathbb{N}\colon\ -x+B\in q\}$. Given $x\in B$ and $m\in\mathbb{N}$ such that $8^m>x$, one has $\mathbb{N}8^m\in q$ and $\mathbb{N}8^m\subseteq -x+B$. □

Lemma 4.7. *Let $i\in\{3,5\}$ and let A be an infinite subset of ω. Then $\{p\in\mathbb{H}\colon\ \tilde{f}_A(p)=i\}$ is a right ideal of $\mathbb{H}$ and $\{p\in\mathbb{H}\colon\ \tilde{g}_A(p)=i\}$ is a left ideal of $\mathbb{H}$.*

Proof. By Lemma 4.6, both of these sets are nonempty. Let $p\in\mathbb{H}$ such that $\tilde{f}_A(p)=i$ and let $q\in\mathbb{H}$. Then $f_A^{-1}[\{i\}]\in p$. To see that $f_A^{-1}[\{i\}]\in p+q$ we show that $f_A^{-1}[\{i\}]\subseteq\{x\in\mathbb{N}\colon\ -x+f_A^{-1}[\{i\}]\in q\}$. So let $x\in f_A^{-1}[\{i\}]$ and pick $m\in\mathbb{N}$ such that $8^m>x$. Then $\mathbb{N}8^m\in q$ and $\mathbb{N}8^m\subseteq -x+f_A^{-1}[\{i\}]$.

Now let $p\in\mathbb{H}$ such that $\tilde{g}_A(p)=i$ and let $q\in\mathbb{H}$. Then $g_A^{-1}[\{i\}]\in p$. To see that $g_A^{-1}[\{i\}]\in q+p$ we show that for all $x\in\mathbb{N}$, $-x+g_A^{-1}[\{i\}]\in q$. So let $x\in\mathbb{N}$ and pick $m\in\mathbb{N}$ such that $8^m>x$. Then $g_A^{-1}[\{i\}]\cap\mathbb{N}8^m\in q$ and $g_A^{-1}[\{i\}]\cap\mathbb{N}8^m\subseteq -x+g_A^{-1}[\{i\}]$. □

Lemma 4.8. *Let A and B be infinite subsets of $\mathbb{N}$ for which $|A\cap B|<\omega$. Then, for any $q\in E(K(\beta\mathbb{N}))$, $\tilde{h}_A(q)$ and $\tilde{h}_B(q)$ belong to different minimal right ideals and to different minimal left ideals of $\beta\mathbb{N}$.*

Proof. For $i\in\{3,5\}$, let $i'=8-i$. Let $q\in E(K(\beta\mathbb{N}))$. Then $q\in\mathbb{H}$ so $q\in K(\mathbb{H})$. By Lemma 4.6, $\tilde{f}_A(q)\in\{3,5\}$ and $\tilde{g}_A(q)\in\{3,5\}$. Let $i=\tilde{f}_A(q)$ and let $j=\tilde{g}_A(q)$. Then $f_A^{-1}[\{i\}]\in q$ and $g_A^{-1}[\{j\}]\in q$. For $x\in f_A^{-1}[\{i\}]$, $f_A(h_A(x))=i'$ and for $x\in g_A^{-1}[\{j\}]$, $g_A(h_A(x))=j'$. So $\tilde{f}_A(\tilde{h}_A(q))=i'$ and $\tilde{g}_A(\tilde{h}_A(q))=j'$.

Let $m>\max(A\cap B)$. For $x\in\mathbb{N}8^m\cap f_A^{-1}[\{i\}]$, $f_A(h_B(x))=i$ and for $x\in\mathbb{N}8^m\cap g_A^{-1}[\{j\}]$, $g_A(h_B(x))=j$. So $\tilde{f}_A(\tilde{h}_B(q))=i$ and $\tilde{g}_A(\tilde{h}_B(q))=j$.

Thus $\tilde{h}_A(q)$ and $\tilde{h}_B(q)$ belong to different minimal right ideals and to different minimal left ideals of $\mathbb{H}$ and consequently of $\beta\mathbb{N}$. □

Theorem 4.9. *If $\mathcal{M}^*\neq\emptyset$, then*

$$\left|\left\{R\colon\ R \text{ is a minimal right ideal of } \beta\mathbb{N} \text{ and } E(R)\subseteq\mathcal{M}\right\}\right|\geqslant\mathfrak{c}.$$

In fact, there is a set $Q\subseteq\mathcal{M}$ such that $|Q|=\mathfrak{c}$ and any two distinct elements of Q belong to different minimal right ideals and to different minimal left ideals of $\beta\mathbb{N}$.

Proof. By Theorem 4.3, the first conclusion follows from the second.

Assume that $\mathcal{M}^*\neq\emptyset$ and pick by Theorem 4.2 $q\in\mathcal{M}$. Pick $p\in\mathbb{H}$ such that $\Psi_{q,p}$ is not discrete. By Lemma 4.5, for each $A\subseteq\omega$, $\Psi_{\tilde{h}_A(q),\tilde{h}_A(p)}$ is not discrete and $\tilde{h}_A(p)\in\mathbb{H}$ so $\tilde{h}_A(q)\in\mathcal{M}$.

Let $\mathcal{A}$ be an almost disjoint family of $\mathfrak{c}$ infinite subsets of $\mathbb{N}$ and let $Q=\{\tilde{h}_A(q)\colon\ A\in\mathcal{A}\}$. By Lemma 4.8, Q has the required properties. □

By requiring that $p\in\{2^n\colon\ n\in\mathbb{N}\}^*$ rather than just in $\mathbb{H}$, we can strengthen the conclusion of Theorem 4.9 producing many q which are bad for the same p.

Theorem 4.10. *Let $p\in\{2^n\colon\ n\in\mathbb{N}\}^*$ and let*

$$M=\left\{q\in E\big(K(\beta\mathbb{N})\big)\colon\ \Psi_{q,p} \text{ is not discrete}\right\}.$$

If $M\neq\emptyset$, there is a subset Q of M such that $|Q|=\mathfrak{c}$ and any two distinct elements of Q belong to different minimal right ideals and to different minimal left ideals of $\beta\mathbb{N}$.

Proof. We note that $p\in\{i8^n\colon\ n\in\mathbb{N}\}^*$ for some $i\in\{1,2,4\}$. Since h_A is the identity map on $\{i8^n\colon\ n\in\mathbb{N}\}$ if $i\in\{1,2,4\}$, it follows that $\tilde{h}_A(p)=p$ for every $A\subseteq\omega$. So $\tilde{h}_A$ is a continuous isomorphism from $\Psi_{q,p}$ onto $\Psi_{\tilde{h}_A(q),p}$ and $\tilde{h}_A[M]=M$.

As in the previous proof, let $\mathcal{A}$ be an almost disjoint family of $\mathfrak{c}$ infinite subsets of $\mathbb{N}$ and let $Q=\{\tilde{h}_A(q)\colon\ A\in\mathcal{A}\}$. □

We conclude by mentioning some tantalizing open questions.

Question 4.11.

(1) *Do there exist any $p\in\{2^n\colon\ n\in\mathbb{N}\}^*$ and any $q\in E(K(\beta\mathbb{N}))$ for which the subgroup generated by $q+p+q$ is not discrete?*
(2) *Does $\mathbb{H}$ contain any nondiscrete copy of $\mathbb{Z}$?*

(3) *Does $\beta\mathbb{N}$ contain any infinite discrete commutative groups which are not isomorphic to $\mathbb{Z}$?*
(4) *Is there any element $p \in \mathbb{N}^*$ with the property that there is no $q \in E(\beta\mathbb{N})$ for which $q + p + q$ generates a discrete group in $\beta\mathbb{N}$?*

With respect to Question (2), as we noted in the introduction, if q is an idempotent in $\beta\mathbb{Z}$, then $\mathbb{Z} + q$ is a nondiscrete copy of $\mathbb{Z}$, but its intersection with $\mathbb{H}$ is just $\{q\}$. We do know that $\mathbb{H}$ contains nondiscrete groups, since the maximal groups in $\mathbb{H}$ each have $2^{\mathfrak{c}}$ elements (by [3, Corollary 7.36]) and, as is well known, any discrete subset of $\beta\mathbb{N}$ has at most $\mathfrak{c}$ members.

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